

The Minimal Landau Background Gauge on the Lattice

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We present the first numerical implementation of the minimal Landau background gauge for Yang-Mills theory on the lattice. Our approach is a simple generalization of the usual minimal Landau gauge and is formulated for general $SU(N)$ gauge group. We also report on preliminary tests of the method in the four-dimensional $SU(2)$ case, using different background fields. Our tests show that the convergence of the numerical minimization process is comparable to the case of a null background. The uniqueness of the minimizing functional employed is briefly discussed.

In Ref. [1] Cornwall pleaded with the lattice community for an answer to the following question: *Can you find a way of doing lattice simulations in the background-field Feynman gauge?* The reason for this request is that one can show [2] — to all orders in perturbation theory — that there is a simple correspondence between the background-field method in the Feynman gauge [3] and the so-called pinch technique [4], which allows one to build gauge-invariant off-shell Green functions in the continuum.

Let us note that the numerical implementation of Landau gauge fixing (e.g. for the evaluation of Green functions) is well understood [5]. Recently, it has been shown that practical simulations of the linear covariant gauge are also possible [6] and that, with a suitable discretization of the gluon field, it becomes feasible to treat the Feynman gauge [7]. Here we present the first numerical implementation of the minimal Landau background gauge on the lattice. Our proposal is based on Ref. [8], which considers this gauge in the continuum.

The covariant background gauge condition is introduced [9] by splitting the (continuum) Yang-Mills field $A_\mu(x)$ into a quantum fluctuation component $Q_\mu(x)$ and a background field $B_\mu(x)$, i.e.

$$A_\mu(x) = Q_\mu(x) + B_\mu(x), \quad (1)$$

where $A_\mu(x)$ is given in terms of the generators T_b of the $SU(N)$ gauge group by $A_\mu(x) = A_\mu^b(x) T_b$ [and similarly for $Q_\mu(x)$ and $B_\mu(x)$]. Note that $B_\mu(x)$ is in principle arbitrary [10]. Then, the usual covariant gauge condition

$$\partial_\mu A_\mu(x) = \Lambda(x) = \Lambda^b(x) T_b \quad (2)$$

becomes

$$\partial_\mu Q_\mu(x) + i [B_\mu(x), Q_\mu(x)] \equiv D_\mu[B] Q_\mu(x) = \Lambda(x). \quad (3)$$

Here $D_\mu[B]$ is the background-field covariant derivative and $\Lambda^b(x)$ is a Gaussian-distributed real variable. Clearly, for a null background field $B_\mu(x) = 0$ one has $Q_\mu(x) = A_\mu(x)$ and the usual covariant gauge condition (2) is recovered. For $\Lambda(x) = 0$ the gauge condition (3) is the Landau background gauge condition.

Let us recall that the continuum gauge transformation of the Yang-Mills field, i.e.

$$A_\mu^{(g)}(x) = g(x) A_\mu(x) g^\dagger(x) - i g(x) \partial_\mu g^\dagger(x), \quad (4)$$

becomes

$$A_\mu^{(g)}(x) \approx A_\mu(x) + D_\mu[A] \gamma(x) \quad (5)$$

if an infinitesimal gauge transformation

$$g(x) = \exp[-i \gamma(x)] \approx 1 - i \gamma(x) \quad (6)$$

is considered, where $\gamma(x) = \gamma^b(x) T_b$. [Note that, with our notation, the generators T_b are Hermitian. In what follows we will also employ the relations $\text{Tr } T_b = 0$ and $\text{Tr } \{T_b T_c\} \propto \delta_{bc}$.] Then, using the splitting in Eq. (1), there is clearly no unique way of defining the infinitesimal gauge transformations $Q_\mu^{(g)}(x)$ and $B_\mu^{(g)}(x)$ for the quantum fluctuation and the background fields. Indeed, depending on which of the three terms $\partial_\mu \gamma(x)$, $i [Q_\mu(x), \gamma(x)]$ and $i [B_\mu(x), \gamma(x)]$ [see Eq. (5)] are included in $Q_\mu^{(g)}(x)$ and $B_\mu^{(g)}(x)$, eight different sets of gauge transformations arise naturally. Among these, two common choices are

$$Q_\mu^{(g)}(x) = Q_\mu(x) + D_\mu[B] \gamma(x) + i [Q_\mu(x), \gamma(x)] \quad (7)$$

$$B_\mu^{(g)}(x) = B_\mu(x) \quad (8)$$

and

$$Q_\mu^{(g)}(x) = Q_\mu(x) + i [Q_\mu(x), \gamma(x)] \quad (9)$$

$$B_\mu^{(g)}(x) = B_\mu(x) + D_\mu[B] \gamma(x). \quad (10)$$

These two transformations are referred to [11] as the quantum transformation and the background transformation, respectively.

The minimal Landau gauge (in the continuum) is obtained [12] by considering stationary points of the minimizing functional

$$\mathcal{E}[A, g] = \int d^d x \operatorname{Tr} \left\{ A_\mu^{(g)}(x) A_\mu^{(g)}(x) \right\} . \quad (11)$$

Indeed, the first variation with respect to the gauge transformation $g(x)$ gives

$$\begin{aligned} \mathcal{E}[A, g] &\approx \mathcal{E}[A, 1] + 2 \int d^d x \operatorname{Tr} \left\{ A_\mu(x) D_\mu[A] \gamma(x) \right\} \\ &= \mathcal{E}[A, 1] - 2 \int d^d x \operatorname{Tr} \left\{ \gamma(x) \partial_\mu A_\mu(x) \right\} , \end{aligned} \quad (12)$$

where we used Eq. (5), the relation

$$\operatorname{Tr} \left\{ A_\mu(x) [A_\mu(x), \gamma(x)] \right\} = 0 \quad (13)$$

and integration by parts. (As is usually done, we make the assumption that the boundary term in the integration by parts gives a null contribution.) Thus, a stationary point of the functional (11) satisfies the condition

$$\operatorname{Tr} \left\{ T_b \partial_\mu A_\mu(x) \right\} = 0 , \quad (14)$$

which is equivalent to Eq. (2) for $\Lambda(x) = 0$.

Working in a similar way, one can also obtain the minimal Landau background gauge. Indeed, the minimization of the functional [8]

$$\mathcal{E}[Q, g] = \int d^d x \operatorname{Tr} \left\{ Q_\mu^{(g)}(x) Q_\mu^{(g)}(x) \right\} , \quad (15)$$

yields the variation

$$\begin{aligned} \mathcal{E}[Q, g] &\approx \mathcal{E}[Q, 1] + 2 \int d^d x \operatorname{Tr} \left\{ Q_\mu(x) D_\mu[B] \gamma(x) \right. \\ &\quad \left. + i Q_\mu(x) [Q_\mu(x), \gamma(x)] \right\} , \end{aligned} \quad (16)$$

if we use the gauge transformation (7). The above expression may be written as

$$\mathcal{E}[Q, g] \approx \mathcal{E}[Q, 1] - 2 \int d^d x \operatorname{Tr} \left\{ \gamma(x) D_\mu[B] Q_\mu(x) \right\} \quad (17)$$

if we again integrate by parts, use Eq. (13) and note the relation

$$\operatorname{Tr} \left\{ Q_\mu(x) [B_\mu(x), \gamma(x)] \right\} = -\operatorname{Tr} \left\{ \gamma(x) [B_\mu(x), Q_\mu(x)] \right\} . \quad (18)$$

Thus, in this case, the stationarity condition implies the gauge-fixing relation

$$\operatorname{Tr} \left\{ T_b D_\mu[B] Q_\mu(x) \right\} = 0 , \quad (19)$$

which is equivalent to Eq. (3) for $\Lambda(x) = 0$. Clearly, for a null background, i.e. $B_\mu(x) = 0$, the minimizing functional (15) coincides with the usual Landau-gauge functional (11) and the gauge condition (14) is recovered.

More in general one should note that, by considering quadratic terms in $Q_\mu(x)$ and $B_\mu(x)$, there are only three terms that can contribute to the minimizing functional of the minimal Landau background gauge, i.e. $Q_\mu(x) Q_\mu(x)$, $Q_\mu(x) B_\mu(x)$ and $B_\mu(x) B_\mu(x)$. However, if one wants to obtain the minimal Landau-gauge functional (11) in the limit $B_\mu(x) \rightarrow 0$, then the minimizing functional $\mathcal{E}[Q, g]$ in Eq. (15) is the only choice at our disposal. In this sense, the minimizing functional $\mathcal{E}[Q, g]$ is *unique*. Moreover, of the eight natural sets of gauge transformations for the quantum field and the background field (see discussion above), one can verify that only the quantum transformation (7)–(8) and the set

$$Q_\mu^{(g)}(x) = Q_\mu(x) + D_\mu[B] \gamma(x) \quad (20)$$

$$B_\mu^{(g)}(x) = B_\mu(x) + i [Q_\mu(x), \gamma(x)] \quad (21)$$

yield the gauge condition (19). Of course, if one lifts the requirement of recovering the functional (11) for $B_\mu(x) = 0$, then the minimal background Landau gauge can also be implemented by considering for example the minimizing functional $\int d^d x \operatorname{Tr} \left\{ Q_\mu^{(g)}(x) B_\mu^{(g)}(x) \right\}$ with the gauge transformation $Q_\mu^{(g)}(x) = Q_\mu(x)$ and $B_\mu^{(g)}(x) = B_\mu(x) + D_\mu[B] \gamma(x) + i [Q_\mu(x), \gamma(x)]$.

The above results may be easily extended to the lattice formulation of Yang-Mills theories. To this end, we write the link variables entering the lattice action as [13]

$$U_\mu(x) = W_\mu(x) V_\mu(x) . \quad (22)$$

We also set

$$U_\mu(x) = \exp [i a A_\mu(x)] \quad (23)$$

$$W_\mu(x) = \exp [i a Q_\mu(x)] \quad (24)$$

$$V_\mu(x) = \exp [i a B_\mu(x)] , \quad (25)$$

where a is the lattice spacing. At the same time, we define [14]

$$A_\mu(x) = \frac{U_\mu(x) - U_\mu^\dagger(x)}{2ia} \Big|_{\text{traceless}} , \quad (26)$$

and similarly for $Q_\mu(x)$ and $B_\mu(x)$. Then, Eq. (1) is immediately recovered, modulo discretization effects.

The lattice gauge transformation

$$U_\mu^{(g)}(x) = g(x) U_\mu(x) g^\dagger(x + ae_\mu) \quad (27)$$

can also be split among the quantum link $W_\mu(x)$ and the background link $V_\mu(x)$. For example, the quantum transformation (7)–(8) is obtained by considering

$$W_\mu^{(g)}(x) = g(x) W_\mu(x) V_\mu(x) g^\dagger(x + ae_\mu) V_\mu^\dagger(x) \quad (28)$$

$$V_\mu^{(g)}(x) = V_\mu(x), \quad (29)$$

while for the background transformation (9)–(10) we have

$$W_\mu^{(g)}(x) = g(x) W_\mu(x) g^\dagger(x) \quad (30)$$

$$V_\mu^{(g)}(x) = g(x) V_\mu(x) g^\dagger(x + ae_\mu). \quad (31)$$

Clearly, in both cases the link variable $U_\mu(x)$ transforms as in Eq. (27). Moreover, using Eqs. (23)–(25) and the lattice definitions of the fields $A_\mu(x)$, $Q_\mu(x)$ and $B_\mu(x)$ in terms of the link variables $U_\mu(x)$, $W_\mu(x)$ and $V_\mu(x)$, one recovers Eqs. (7)–(10) when an infinitesimal gauge transformation (6) is considered. For example, Eq. (28) gives

$$W_\mu^{(g)}(x) \approx [1 - i\gamma(x)] [1 + iaQ_\mu(x)] [1 + iaB_\mu(x)] [1 + i\gamma(x + ae_\mu)] [1 - iaB_\mu(x)] \quad (32)$$

$$\approx 1 + ia \left\{ \partial_\mu \gamma(x) + Q_\mu(x) + i[Q_\mu(x), \gamma(x)] + i[B_\mu(x), \gamma(x)] \right\} = 1 + iaQ_\mu^{(g)}(x), \quad (33)$$

in agreement with Eq. (7).

One can also define a minimizing functional for the Landau background gauge on the lattice. Indeed, in the limit of small lattice spacing a , the functional

$$\mathcal{E}[W, g] = - \sum_{x, \mu} \Re \text{Tr} W_\mu^{(g)}(x) \quad (34)$$

is equivalent to

$$\mathcal{E}[W, g] \approx a^2 \sum_{x, \mu} \text{Tr} \left\{ Q_\mu^{(g)}(x) Q_\mu^{(g)}(x) \right\}, \quad (35)$$

modulo constant terms. (Here we use \Re to indicate the real part.) At the same time, for $V_\mu(x) = 1$ and $W_\mu^{(g)}(x) = U_\mu^{(g)}(x)$ we recover the usual minimizing func-

tional for the Landau-gauge condition [5]

$$\mathcal{E}[U, g] = - \sum_{x, \mu} \Re \text{Tr} \left\{ g(x) U_\mu(x) g(x + ae_\mu) \right\}. \quad (36)$$

Also, if $W_\mu^{(g)}(x)$ transforms as in Eq. (28) and we consider an infinitesimal gauge transformation (6) we find

$$\begin{aligned} \mathcal{E}[W, g] \approx \mathcal{E}[W, 1] - i \sum_{x, \mu} \Im \text{Tr} \left\{ \gamma(x) [U_\mu(x) V_\mu^\dagger(x) \right. \\ \left. - V_\mu^\dagger(x - ae_\mu) U_\mu(x - ae_\mu)] \right\}, \end{aligned} \quad (37)$$

where \Im indicates the imaginary part. As a consequence, a stationary point of the minimizing functional (34) implies the gauge condition

$$= \text{Tr} \left\{ T_b \sum_\mu [W_\mu(x) - W_\mu^\dagger(x) - V_\mu^\dagger(x - ae_\mu) U_\mu(x - ae_\mu) + U_\mu^\dagger(x - ae_\mu) V_\mu(x - ae_\mu)] \right\}, \quad (38)$$

where we used the Hermiticity of the generators T_b . Finally, by adding and subtracting $\text{Tr} \{ T_b \sum_\mu [W_\mu(x - ae_\mu) - W_\mu^\dagger(x - ae_\mu)] \}$ we find that the null quantity in the above equation can be written conveniently as the sum of two terms. The first one is taken as $\text{Tr} \{ T_b \sum_\mu [W_\mu(x) - W_\mu^\dagger(x) - W_\mu(x - ae_\mu) + W_\mu^\dagger(x - ae_\mu)] \}$ and is equal (at leading order in the lattice spacing a) to

$$2ia \text{Tr} \left\{ T_b \sum_\mu [Q_\mu(x) - Q_\mu(x - ae_\mu)] \right\} \approx 2ia^2 \text{Tr} \left\{ T_b \sum_\mu \partial_\mu Q_\mu(x) \right\}. \quad (39)$$

The second term is then given by

$$\begin{aligned} \text{Tr} \Big\{ T_b \sum_{\mu} [U_{\mu}(x - ae_{\mu}) V_{\mu}^{\dagger}(x - ae_{\mu}) + U_{\mu}^{\dagger}(x - ae_{\mu}) V_{\mu}(x - ae_{\mu}) \\ - V_{\mu}^{\dagger}(x - ae_{\mu}) U_{\mu}(x - ae_{\mu}) - V_{\mu}(x - ae_{\mu}) U_{\mu}^{\dagger}(x - ae_{\mu})] \Big\}. \end{aligned} \quad (40)$$

Note that for a null background field, i.e. $B_{\mu}(x) = 0$ and $V_{\mu}(x) = 1$, the quantity above is identically zero. In this case, we have $Q_{\mu}(x) = A_{\mu}(x)$ [i.e. $W_{\mu}(x) = U_{\mu}(x)$] and the gauge condition (38) becomes [see also Eq. (39)] the usual lattice Landau-gauge condition $\text{Tr} \{ T_b \sum_{\mu} [A_{\mu}(x) - A_{\mu}(x - ae_{\mu})] \} = 0$. In the $B_{\mu}(x) \neq 0$ case and in the limit of small lattice spacing a , one can check that the quantity (40) is, at leading order, equal to the expression

$$-2a^2 \text{Tr} \{ T_b \sum_{\mu} [B_{\mu}(x), Q_{\mu}(x)] \}. \quad (41)$$

Thus, the stationarity condition (38) implies (again at leading order in a)

$$\text{Tr} \left\{ T_b \sum_{\mu} \partial_{\mu} Q_{\mu}(x) + i [B_{\mu}(x), Q_{\mu}(x)] \right\} = 0, \quad (42)$$

in agreement with Eq. (19).

As discussed above, given a fixed lattice configuration $\{U_{\mu}(x)\}$, the usual minimal Landau gauge may be imposed by numerically minimizing the functional (36). In particular, by considering local updates for the gauge-fixing transformation $\{g(x)\}$ it is easy to verify that, for a given site y , the contribution of $g(y)$ to the minimizing functional may be written as [15]

$$\mathcal{E}[U, g] = \text{constant} + \Re \text{Tr} \{ g(y) h(x) \} \quad (43)$$

with $h(x) = \sum_{\mu} [U_{\mu}(x) + U_{\mu}^{\dagger}(x - ae_{\mu}) + U_{\mu}^{\dagger}(x) + U_{\mu}(x - ae_{\mu})]$. Then, different gauge-fixing algorithms correspond to different choices for the iterative updates of the gauge transformation $g(y)$ in Eq. (43).

In the case of the minimal Landau background gauge, one can consider the minimizing functional $\mathcal{E}[W, g]$, defined in Eqs. (34) and e.g. (28), where $\{W_{\mu}(x)\}$ and $\{V_{\mu}(x)\}$ are given (i.e. fixed) quantum and background configurations respectively. It is important to stress that also in this case the contribution of $g(y)$ to the minimizing functional $\mathcal{E}[W, g]$ may be written as in Eq. (43). In this case, the quantity $h(x)$ is equal to

$$\begin{aligned} h(x) = \sum_{\mu} \left[W_{\mu}(x) + U_{\mu}^{\dagger}(x - ae_{\mu}) V_{\mu}(x - ae_{\mu}) \right. \\ \left. + W_{\mu}^{\dagger}(x) + V_{\mu}^{\dagger}(x - ae_{\mu}) U_{\mu}(x - ae_{\mu}) \right]. \end{aligned} \quad (44)$$

Thus, all formulae used for the minimal Landau background gauge are natural generalizations of the formulae used for the usual minimal Landau gauge. This implies that, at least for sufficiently smooth background configurations $\{V_{\mu}(x)\}$, we should expect similar convergence of the gauge-fixing algorithms for these two gauge-fixing conditions.

In order to verify this, we have carried out some tests in the SU(2) case, considering lattice volumes $V = 8^4$ and $V = 16^4$ with a lattice coupling $\beta = 2.2$, corresponding to a lattice spacing a of about 0.210 fermi. This means that the thermalized configurations $\{U_{\mu}(x)\}$ are reasonably “rough” and provide a good test for the gauge-fixing algorithm employed. For the background-field $\{V_{\mu}(x)\}$ we have considered three types of configurations with three setups each, namely [here, σ_j are the three Pauli matrices, with σ_3 being the diagonal one]:

- a) **random center configuration** (RCC) $V_{\mu}(x) = \pm 1$, which can be interpreted as a random configuration of *thin* vortices [16], with, on average, 10%, 30% or 50% of the links equal to -1 ;
- b) **random Abelian configuration** (RAC) $V_{\mu}(x) = \exp[i\theta(x)\sigma_3]$, which may be interpreted as a random configuration of Abelian monopoles [17], with the angle $\theta(x)$ uniformly distributed in the interval $[0, 2\pi f]$ and f equal to 0.1, 0.3 or 0.5;
- c) **super-instanton configuration** (SIC) [18] given by $V_2(x) = \exp[i c \min(x_1, N - x_1) \sum_j \sigma_j / \sqrt{3N^2}]$ and $V_{\mu}(x) = 0$ otherwise, with $c = 0.01, 0.05$ or 0.1 , where N is the number of lattice sites per direction.

For the two lattice volumes above, we consider ten gauge-field configurations and, in each case, we fix the minimal background Landau gauge, using the stochastic-overrelaxation algorithm [15], for the nine choices of background fields described above. The number of minimizing sweeps necessary to achieve the prescribed accuracy was then compared to that used in the case of a null background (i.e. Landau gauge). Here we stop the gauge-fixing algorithm when the average magnitude squared of the quantity on the r.h.s. of Eq. (38) is smaller than 10^{-14} . Note that we tuned the stochastic-overrelaxation

$B_\mu(x)$	8^4			16^4		
	aver.	min.	max.	aver.	min.	max.
null background	217	190	290	508	396	773
RCC 10%	348	190	685	976	503	1729
RCC 30%	624	342	1391	1344	818	1979
RCC 50%	647	444	1032	1711	1002	2714
RAC $f = 0.1$	224	191	323	677	417	1226
RAC $f = 0.3$	326	190	1112	582	436	967
RAC $f = 0.5$	401	279	595	813	494	1495
SIC $c = 0.01$	637	372	855	1852	1238	3503
SIC $c = 0.05$	188	172	256	520	344	808
SIC $c = 0.1$	177	170	203	365	343	430

TABLE I. Average, minimum and maximum number of sweeps necessary to achieve the prescribed accuracy for the two lattice volumes and for the nine different background fields considered in our tests (see description in the text). For a comparison, we also include the case of a null background.

algorithm in the case of a null background, setting the parameter p of the algorithm (see [15]) equal to 0.83 for $V = 8^4$ and to 0.91 for $V = 16^4$. The same setup was then used for non-zero backgrounds. Results of these tests are shown in Table I. One sees that the convergence of the gauge-fixing algorithm for a non-zero back-

ground is indeed similar to the case of the usual minimal Landau gauge. Of course, by tuning the parameter p also in the general case, one can improve the results. In fact, e.g. for $V = 8^4$ and background RCC 30%, we find that with $p = 0.92$ the number of sweeps decreases considerably, being between 418 and 653, with an average value of about 460. Similarly, for $V = 16^4$ and the SIC background with $c = 0.01$, we obtain for $p = 0.96$ that the number of sweeps is between 794 and 1934, with an average value of about 1001.

The above results indicate that numerical simulations in the minimal Landau background gauge are indeed feasible. One should also stress that the extension of the method presented here to the case of the minimal covariant background gauge is, in principle, straightforward [6]. This extension, as well as the numerical evaluation of Green functions in minimal Landau background gauge, is postponed to future studies.

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- [1] J. M. Cornwall, PoS **QCD-TNT09**, 007 (2009).
[2] D. Binosi and J. Papavassiliou, Phys. Rev. **D66**, 111901 (2002).
[3] R. F. Dashen and D. J. Gross, Phys. Rev. **D23**, 2340 (1981), also in *Lattice gauge theories and Monte Carlo simulations*, C. Rebbi (World Scientific Pub. Co., 1983).
[4] J. M. Cornwall, Phys. Rev. **D26**, 1453 (1982).
[5] See for example Section 3 in L. Giusti *et al.*, Int. J. Mod. Phys. **A16**, 3487 (2001).
[6] A. Cucchieri, T. Mendes and E. M. S. Santos, Phys. Rev. Lett. **103**, 141602 (2009).
[7] A. Cucchieri, T. Mendes, G. M. Nakamura and E. M. S. Santos, AIP Conf. Proc. **1354**, 45 (2011); *ibid.* PoS **FACESQCD**, 026 (2010).
[8] D. Zwanziger, Nucl. Phys. **B209**, 336 (1982).
[9] See for example Section 16.6 in *An Introduction To Quantum Field Theory*, M. E. Peskin, D. V. Schroeder (Addison-Wesley Pub. Co., 1995).
[10] L. F. Abbott, M. T. Grisaru and R. K. Schaefer, Nucl. Phys. **B229**, 372 (1983); M. Luscher and P. Weisz, Nucl. Phys. **B452**, 213 (1995).
[11] See e.g. Section 8.2 in *Gauge Field Theories*, S. Pokorski (Cambridge University Press, second edition, 2000).
[12] See for example Section 2.2.1 in N. Vandersickel and D. Zwanziger, arXiv:1202.1491 [hep-th].
[13] This is a natural definition of a background field configuration on the lattice [see for example P. Cea and L. Cosmai, Phys. Lett. **B264**, 415 (1991)] but, of course, other discretizations are possible.
[14] In order to reduce discretization effects — see for example D. B. Leinweber, J. I. Skullerud, A. G. Williams and C. Parrinello [UKQCD Collaboration], Phys. Rev. **D60**, 094507 (1999) [Erratum-ibid. **D61**, 079901 (2000)] — one should define the r.h.s. of Eq. (26) equal to $A_\mu(x+ae_\mu/2)$, where e_μ is a unit vector in the positive μ direction, instead of $A_\mu(x)$. However, since the leading order results coincide in the two cases, here we prefer to simplify the notation and use the definition (26).
[15] A. Cucchieri and T. Mendes, Nucl. Phys. **B471**, 263 (1996).
[16] See e.g. J. Greensite, Lect. Notes Phys. **821**, 1 (2011).
[17] See for example M. N. Chernodub and M. I. Polikarpov, In *Cambridge 1997, Confinement, duality, and nonperturbative aspects of QCD*, 387.
[18] A. Patrascioiu and E. Seiler, Phys. Rev. Lett. **74**, 1924 (1995).